Control of Distributed Parameter Systems by Moving Force Actuators

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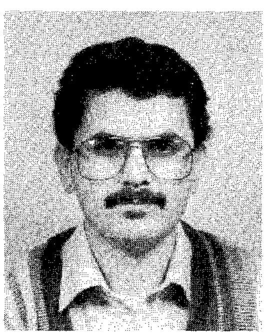
The control of vibrations in flexible structures by moving force actuators is considered. The elimination of a predetermined set of vibratory modes by one or more moving force actuators is shown possible. Two such control strategies are proposed. The first strategy uses a finite number of distinct moving actuators to control an equal number of critical modes with negligible control spillover into the residual modes. The second uses only a single actuator to control a finite number of critical modes; and it is based on energy functions, where the spatial variation of the actuator has to be carefully chosen to guarantee stability of the closed-loop system. Both of these can employ closed-loop strategies for force control. In addition to feedback, the first strategy may also utilize an open-loop force control scheme, such as minimum energy control. Moving actuators need relatively smaller magnitudes of force for accomplishing performance guarantees that are achievable with fixed actuators having larger magnitudes of force.

I. Introduction

It is well known that with a distributed force the control spillover can be eliminated when controlling distributed parameter systems. However, physical realization of such forces controlling distributed parameter systems is difficult. Instead, most control applications are based on point force actuators. Note that moving a point actuator at infinite speed can be viewed as being equivalent to a distributed load, especially in the case of a one-dimensional structure. But it is impossible to move an actuator at infinite speed. The question then is whether or not it is possible to control distributed parameter systems by using actuators that move at finite speeds along the structure.

The use of moving actuators in the control of flexible structures is a relatively unexplored research area. The dynamics of flexible structures with impulses delivered at moving stations has been studied by Bamberger et al.[1] They studied the effect of moving force impulses on the dynamics of a vibrating string. A stabilizing feedback control law[2] was constructed for the vibrating string. A control strategy that suppresses a set of vibratory modes of a simply supported beam by a single actuator moving at a constant speed has also been developed.[3]

Moving actuators can travel randomly in the domain of a flexible structure, or they can be moving electronic probes actuating orthogonal to the structure. Another concept similar to moving actuators is the notion of moving actuation. In



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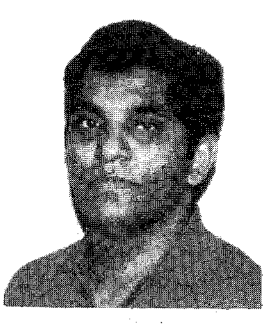

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moving actuation, an assemblage of a set of discrete fixed force actuators can be activated sequentially over different finite time intervals.[4]

There is a considerable body of research on the control of flexible structures by fixed force actuators. The use of such actuators in the active control of vibrations in flexible structures gives acceptable performance only if the modal content of the unwanted motion to be suppressed aligns well with the modes retained in the controller synthesis.[4] For instance, in a truncated model, it is impossible to suppress an initially active mode if that mode is not retained in the model. The response of a distributed parameter system when controlled by fixed actuators is determined by the control forces alone, whereas, in the case of moving actuators, the system's performance will be determined by both the control forces and the spatial locations of the actuators.

The paper is organized as follows. The equations of motion of distributed parameter systems controlled by moving actuators are given in Sec. II. The objectives of using moving actuators are stated in Sec. III. The control of a finite number of modes by the same number of actuators moving at arbitrary speeds is considered in Sec. IV. Comparisons between fixed and moving actuators for the finite time settling control of a finite number of modes by an equal number of actuators are discussed in Sec. V. In Sec. VI, the control of a finite set of modes by a single actuator moving at an arbitrary speed is developed using energy functions. Conclusions are in Sec. VII.

II. Equations of Motion

Consider a distributed parameter system described by

$$L[u(P,t)] + D\left[\frac{\partial u}{\partial t}(P,t)\right] + M\left[\frac{\partial^2 u}{\partial t^2}(P,t)\right] = F(P,t) \quad (1)$$

$$y(P,t) = \left[u(P,t), \frac{\partial u}{\partial t}(P,t)\right]^T \quad (2)$$

$$B_i[u(P,t)] = 0 \qquad i = 1, 2, \ldots \quad (3)$$

where L, D, and M are linear operators characterizing the stiffness, damping, and mass of the system; $u(P,t)$ is the displacement field; P is a point in the system; t is time; $F(P,t)$ is a distributed load; $y(P,t)$ is an output vector of the distributed state, and $B_i = 1, 2, \ldots$, is a set of boundary condition operators. $F(P,t)$ characterizes a finite set of moving point force actuators described by

$$F(P,t) = \sum_{j=1}^{m} F_j(t)\delta[P - P_j(t)] \quad (4)$$

where m is the number of moving actuators; $F_j(t)$ is the corresponding magnitude of the jth actuator; δ is the Dirac delta function, and $P_j(t)$ is the corresponding time-varying spatial location of the jth actuator. Note that, in general, the dynamics of the moving actuators should be included with the dynamics of the distributed parameter system. But, for the purposes of this paper, we assume that the dynamics of the moving actuators are independent of that of the flexible structure. Thus, independent controllers can be developed for each moving actuator. Assume that the operators L, D, and M in Eq. (1) are self-adjoint, and D is a proportional damping operator satisfying

$$D = \alpha L + \beta M \quad (5)$$

where α and β are constants. Thus, the modal equations can be written as

$$\ddot{\eta}_i(t) + 2\zeta_i w_i \dot{\eta}_i(t) + w_i^2 \eta_i(t) = f_i(t) \quad (6)$$

where w_i are the natural frequencies, and where

$$\zeta_i = (1/2w_i)\,[\alpha w_i^2 + \beta] \quad (7)$$

are the damping ratios and

$$f_i(t) = \sum_{j=1}^{m} M\{\phi_i[P_j(t)]\}F_j(t) \quad (8)$$

are the modal forces and $\phi_i[P_j(t)]$ is the ith mode evaluated at the location $P_j(t)$ of the jth actuator.

III. Objectives

The objectives of using moving actuators at finite speeds can be stated as the following:

1) Good approximation of distributed loads for the reduction of energy flow into the residual modes.

2) Finite time settling control of vibratory motion with smaller control forces than are needed with fixed actuators.

Three different control strategies for moving actuators are considered. The first (considered in Sec. IV) uses as many actuators as the number of controlled modes where the velocity profiles of the moving actuators are arbitrary to the extent that simultaneous crossings of actuators at one or more nodes of any mode are not allowable. The second (Sec. V) is used to determine the minimum energy control with constant velocity profiles. The third control strategy (Sec. VI) emphasizes the use of a single actuator, with the velocity profile determined by Lyapunov energy functions.

IV. Control of a Finite Set of Modes by a Finite Set of Moving Actuators

Assume that the number of moving actuators is equal to the number of controlled modes. The functions $f_i(t)$, $i = 1, 2, \ldots, m$, are generally complicated functions of time. For instance, if the modes ϕ_i are of a cantilevered beam, the functions $f_i(t)$ include terms of the form $\sin x(t)$ and $\sinh x(t)$, where $x(t)$ is an arbitrary function characterizing the space coordinate. One case where these functions are simple is when $x(t)$ is a linear function of time, i.e., the corresponding velocity is constant. In Jayasuriya and Choura[5] a simply supported beam controlled by a single actuator moving at constant speed was considered. They proposed a control strategy that completely eliminates the odd-numbered modes and any initially active modes at a prespecified finite final time.

In this paper, a general form for velocity profiles is considered. The task is to find a set of simple functions $f_i(t)$, $i = 1, 2, \ldots, m$, that defines the controls $F_i(t)$. Assume that the simple functions $f_i(t)$ can be chosen a priori, then from Eq. (8) the following matrix equation results

$$G(t)F(t) = f(t) \quad (9)$$

where

$$G(t) = \begin{bmatrix} M\{\phi_1[P_1(t)]\} & \cdots & M\{\phi_1[P_m(t)]\} \\ \vdots & \ddots & \vdots \\ M\{\phi_m[P_1(t)]\} & \cdots & M\{\phi_m[P_m(t)]\} \end{bmatrix} \quad (10)$$

$$F(t) = \begin{bmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_m(t) \end{bmatrix} \quad \text{physical force vector} \quad (11)$$

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_m(t) \end{bmatrix} \quad \text{modal force vector} \quad (12)$$

For bounded control functions, we require that the matrix $G(t)$ be nonsingular for all times t. If, for instance, $M\{\phi_i[P_j(t)]\}$, for any $P_j(t)$, $j = 1, 2, \ldots, m$, can be written as

$$M\{\phi_i[P_j(t)]\} = N[P_j(t)]\phi_i[P_j(t)]$$

where N is only a function of $P_j(t)$, then a necessary condition for $G(t)$ to be nonsingular is that $\phi_i[P_j(t)]$, $j = 1, 2, \ldots, m$, do not all vanish simultaneously for each $i = 1, 2, \ldots, m$. Physically, this means that simultaneous crossings of all actuators at one or more nodes of any mode ϕ_i, $i = 1, 2, \ldots, m$ are not allowed. However, this is not the only way $G(t)$ can be singular. If $G(t)$ is nonsingular for all t, then the forcing functions $f_i(t)$ can be picked arbitrarily. Therefore, $f_i(t)$, $i = 1, 2, \ldots, m$, can be feedback controls, minimum energy controls,[6] or others that capture the design specifications. Note that if the vector $f(t)$ is selected to be discontinuous, then the vector $F(t)$ must also be discontinuous and have the same number of discontinuities, provided that the matrix $G(t)$ is continuous.

If $|G(t)|$ does not vanish for any time t, then the equations of motion become

$$\ddot{\eta}_1(t) + 2\zeta_1 w_1\dot{\eta}_1(t) + w_1^2\eta_1(t) = f_1(t)$$

$$\ddot{\eta}_2(t) + 2\zeta_2 w_2\dot{\eta}_2(t) + w_2^2\eta_2(t) = f_2(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\ddot{\eta}_m(t) + 2\zeta_m w_m\dot{\eta}_m(t) + w_m^2\eta_m(t) = f_m(t)$$

$$\ddot{\eta}_i(t) + 2\zeta_i w_i\dot{\eta}_i(t) + w_i^2\eta_i(t) = Q_i(t)G^{-1}(t)f(t)$$

$$i = m+1, m+2, \ldots \qquad (13)$$

where

$$Q_i(t) = \left(M\{\phi_i[P_1(t)]\} \cdots M\{\phi_i[P_m(t)]\}\right) \qquad (14)$$

From Eq. (13), the residual modes, corresponding to $i = m+1, m+2, \ldots$, can be severely affected by control spillover[7] when the forcing function $Q_i(t)G^{-1}(t)f(t)$ has large bounds. Small bounds on $Q_i(t)G^{-1}(t)f(t)$ for all t are then preferred to reduce the effect of energy flow into the uncontrolled modes. Next, an example illustrates two cases where, in one, the control spillover is negligible and, in the other, it is not.

Assume that the first two modes of a simply supported beam are the only initially active modes. The control of both modes by two moving point force actuators consists of bringing the modes to rest without severely affecting the remaining modes. For a simply supported beam, Eqs. (13) can be written as

$$\ddot{q}_1 + w_1^2 q_1 = f_1(t) \qquad (15a)$$

$$\ddot{q}_2 + w_2^2 q_2 = f_2(t) \qquad (15b)$$

$$\ddot{q}_i + w_i^2 q_i = Q_i(t)G^{-1}(t)f(t) \qquad i = 3, 4, \ldots \qquad (15c)$$

where

$$Q_i(t) = \sqrt{\frac{2}{\rho AL}}\left[\sin\frac{i\pi x_1(t)}{L} \quad \sin\frac{i\pi x_2(t)}{L}\right] \qquad (16)$$

$$G(t) = \sqrt{\frac{2}{\rho AL}}\begin{bmatrix} \sin\dfrac{\pi x_1(t)}{L} & \sin\dfrac{\pi x_2(t)}{L} \\ \sin\dfrac{2\pi x_1(t)}{L} & \sin\dfrac{2\pi x_2(t)}{L}\end{bmatrix} \qquad (17)$$

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t)\end{bmatrix} \qquad (18)$$

where $w_i = i^2\pi^2\sqrt{EL/\rho AL^4}$, $i = 1, 2, 3, \ldots$, are the natural frequencies, E is the Young's modulus, ρ is the density, L is the length of the beam, and A is the cross-sectional area.

Remark: The special case of a single moving actuator analyzed in Jayasuriya and Choura[3] can be recovered from Eqs. (15–18) by setting

$$G(t) = \sqrt{\frac{2}{\rho AL}}\sin\frac{\pi x_1(t)}{L}$$

$$Q_i(t) = \sqrt{\frac{2}{\rho AL}}\sin\frac{i\pi x_1(t)}{L}$$

$$f_1(t) = \sqrt{\frac{2}{\rho AL}}f_0\sin\frac{j\pi V_0 t}{L}\sin\frac{\pi x_1(t)}{L}$$

$$f_2(t) = 0$$

$$x_1(t) = V_0 t$$

where j is a free integer, V_0 is the velocity of the moving actuator, and $\sqrt{2/\rho AL}f_0$ is the control magnitude.

For the matrix $G(t)$ in Eq. (17) to be nonsingular, we require that both actuators do not cross the end point at $x(t) = 0$ and $x(t) = L$ at any time t, and do not cross any point $0 < x(t) < L$ simultaneously. Assume that the axial time-varying positions of the two actuators along the beam are described by

$$x_1(t) = (11/20)L(t/t_f) + (L/5) \qquad (19a)$$

$$x_2(t) = (11/20)L(t/t_f) + (L/4) \qquad (19b)$$

where t_f is an arbitrary settling time. Note that both actuators are moving at the same constant velocity $(11L/20t_f)$, but start at different locations; therefore, they cannot simultaneously cross the same point in the time range $t = 0$ to $t = t_f$. Assume that system (15) is precisely known or is certain and the initial conditions are

$$q_1(0) = q_2(0) = \frac{L}{10}, \qquad \dot{q}_1(0) = \dot{q}_2(0) = 0$$

$$q_i(0) = \dot{q}_i(0) = 0 \qquad i = 3, 4, 5, \ldots \qquad (20)$$

As discussed previously, the two controls $f_1(t)$ and $f_2(t)$ can be chosen using any method. A special case is where both controls developed below are of the minimum energy type,[6] an open-loop control. It must be emphasized, however, that it is not necessary for them to be open-loop. For example, an alternative set of force controls can be state feedback schemes based on the Independent Modal Space Control (IMSC) method.[7]

Equations (15a) and (15b) can be written as

$$\dot{x}_j(t) = A_j x_j(t) + B_j f_j(t) \qquad j = 1, 2 \qquad (21)$$

where

$$x_j(t) = \begin{bmatrix} x_{1j}(t) \\ x_{2j}(t)\end{bmatrix}, \quad A_j = \begin{bmatrix} 0 & 1 \\ -w_j^2 & 0\end{bmatrix}, \quad B_j = \begin{bmatrix} 0 \\ 1\end{bmatrix} \qquad (22)$$

and f_j is the minimum energy control corresponding to the jth mode, $j = 1, 2$. It is well known that the minimum energy control for system (21) can be computed[8] as

$$f_j(t) = -B_j^T e^{-A_j^T t}\left[\int_0^{t_f} e^{-A_j\tau}B_jB_j^T e^{-A_j^T\tau}\,d\tau\right]^{-1}x_j(0) \qquad (23)$$

which reduces to

$$f_j = w_j(L/10)\left\{\frac{2w_j\sin w_j t_f\cos[w_j(t-t_f)-(\pi/2)]}{w_j^2t_f^2 - \sin^2 w_j t_f} + \frac{2w_j^2 t_f\sin w_j t}{w_j^2 t_f^2 - \sin^2 w_j t_f}\right\}, \qquad j = 1, 2 \qquad (24)$$

Simulations for the above moving actuators are shown in Figs. 1-6. The modal displacements q_i, $i = 1, 2, 3, 4$, are shown in Figs. 1-4. Since modes 3 and 4 are not controlled, residual vibrations of these modes occur after the final time t_f as a consequence of control spillover, which is negligible in this case. The overall displacement of the beam at $(3L/5)$ is displayed in Fig. 5, and the corresponding controls $F_1(t)$ and $F_2(t)$ in Fig. 6.

The physical parameters of the simply supported beam used in the simulations are given below. The beam is made of aluminum:

density ρ	$= 2710$ kg/m^3
Young's modulus E	$= 71 \times 10^9$ N/m^2
width of beam b	$= 8.467 \times 10^{-4}$ m
height h	$= 1.905 \times 10^{-2}$ m
length L	$= 0.762$ m
cross-sectional area A	$= 1.613 \times 10^{-5}$ m^2
moment of area I	$= 9.6361 \times 10^{-13}$ m^4

Note that the overall response after the final time $t_f = 1$ s is almost zero. Thus, with the displacement of actuators described in Eqs. (19), the control spillover is negligible. If the displacements of the actuators are

$$x_1(t) = (161/180)L(t/t_f) + (L/20) \tag{25a}$$

$$x_2(t) = (161/180)L(t/t_f) + (L/18) \tag{25b}$$

the control spillover, shown in Fig. 7, is not negligible after $t_f = 1$ s. The response in Fig. 6 is different to that in Fig. 7, because the magnitude of $G^{-1}(t)$ is larger in the second case. The large magnitude of $G^{-1}(t)$ is the result of the small distance between the two actuators from $t = 0$ to $t = t_f$, and the proximity of the actuators to the end point of the beam at the beginning and the end of the maneuver. For different displacement profiles such as

$$x_1(t) = (11/20)L \sin(\pi t/2t_f) + (L/5) \tag{26a}$$

$$x_2(t) = (11/20)L \sin(\pi t/2t_f) + (L/4) \tag{26b}$$

the modal displacements of modes 3 and 4 are shown in Figs. 8 and 9. As expected, modes 3 and 4 are affected by spillover; thus, residual vibrations are introduced. The corresponding displacement of the beam at $x = (3L/5)$ and the controls $F_1(t)$ and $F_2(t)$ are displayed in Fig. 10. Note that the controls $f_1(t)$ and $f_2(t)$ and the final time $t_f = 1$ s are kept the same.

V. Minimum Energy Control by Moving Actuators

In previous work,[5] minimum energy controls were implemented with fixed actuators. In this paper, the notion of minimum energy control is used with moving actuators, and is

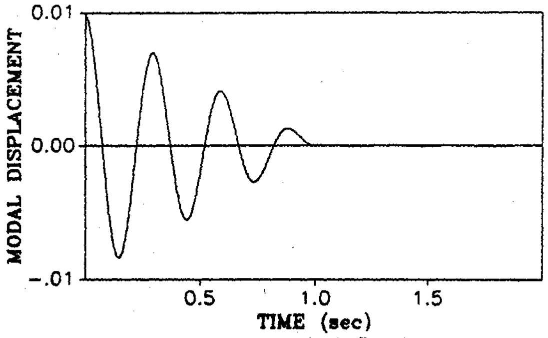

Fig. 1 Displacement of mode 1.

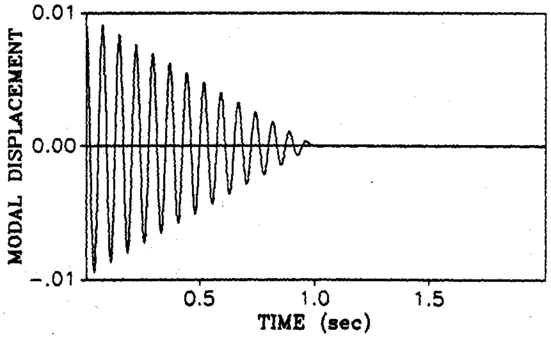

Fig. 2 Displacement of mode 2.

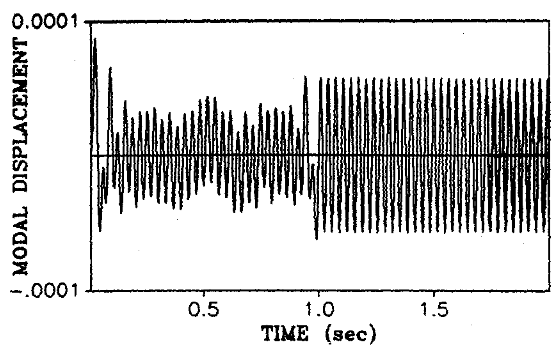

Fig. 3 Displacement of mode 3.

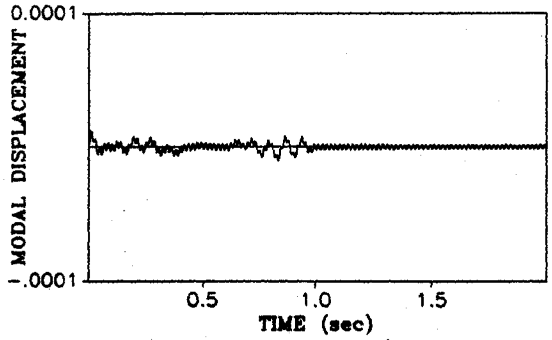

Fig. 4 Displacement of mode 4.

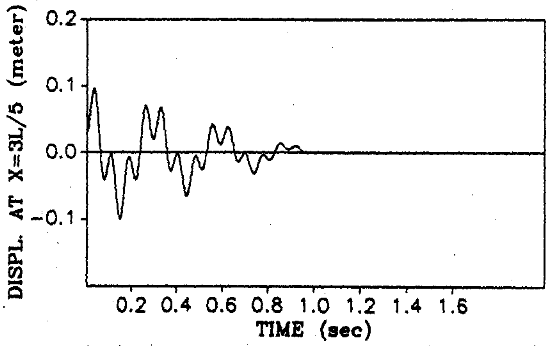

Fig. 5 Beam displacement at $x = (3L/5)$ (two actuators).

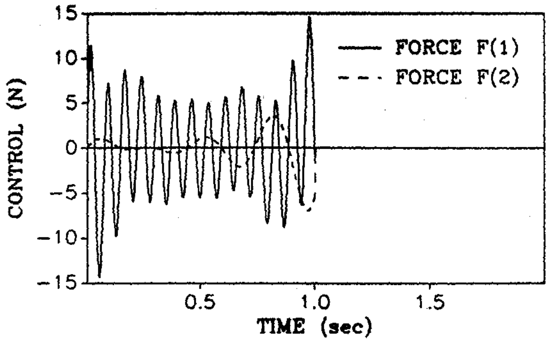

Fig. 6 Controls F_1 and F_2.

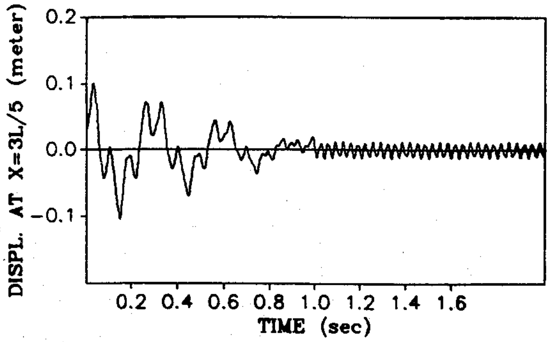

Fig. 7 Beam response with significant control spillover.

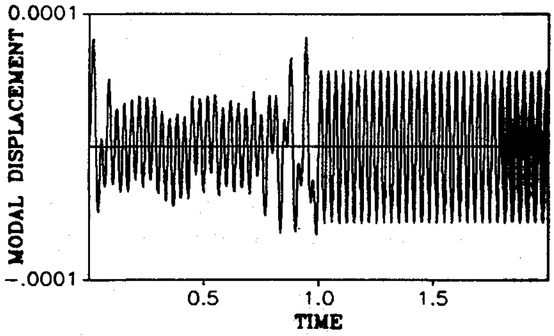

Fig. 8 Displacement of mode 3 (control spillover).

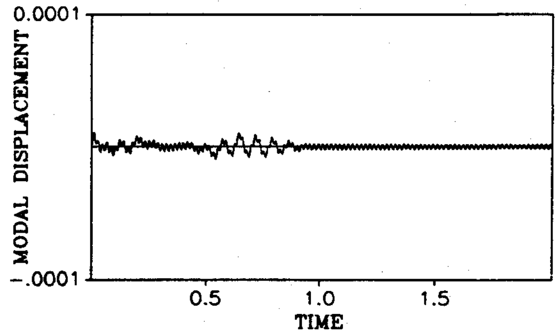

Fig. 9 Displacement of mode 4 (control spillover).

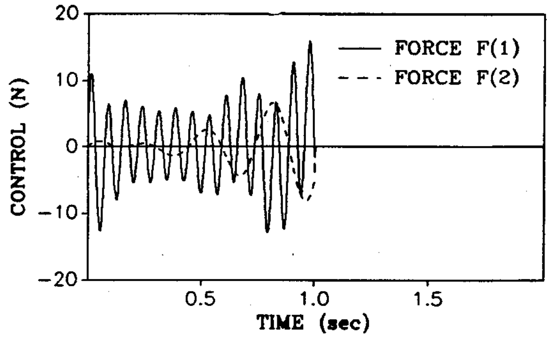

Fig. 10 Controls F_1 and F_2 (control spillover).

compared to that of fixed actuators for a simply supported beam. Let a simply supported beam be described by

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}=\sum_{i=1}^{m}F_i(t)\delta[x-x_i(t)] \qquad 0\le x\le L \qquad t\ge 0 \tag{27}$$

with the boundary conditions

$$v(0,t)=v(L,t)=\frac{\partial^2 v}{\partial x^2}(0,t)=\frac{\partial^2 v}{\partial x^2} \qquad t\ge 0 \tag{28}$$

The natural frequencies of the beam are

$$w_n=(n\pi)^2\sqrt{\frac{EI}{\rho AL^4}} \qquad n=1,2,\ldots \tag{29}$$

and its natural modes are

$$Y_n(x)=\sqrt{\frac{2}{\rho AL}}\sin\frac{n\pi x}{L} \qquad n=1,2,\ldots \tag{30}$$

Substitution of

$$v(x,t)=\sum_{n=1}^{\infty}Y_n(x)q_n(t) \tag{31}$$

in Eq. (27) and the orthogonality property lead to the modal equations

$$\ddot q_n(t)+w_n^2q_n(t)=\sqrt{\frac{2}{\rho AL}}\sum_{i=1}^{m}F_i(t)\sin\frac{n\pi x_i(t)}{L} \tag{32}$$

Assume the magnitudes and the time-varying positions of the moving actuators are expressed as

$$F_i(t)=f_i, \qquad x_i(t)=A_it+B_i, \qquad i=1,2,\ldots,m \tag{33}$$

Substituting Eq. (33) into Eq. (32) leads to

$$\ddot q_n(t)+w_n^2q_n(t)=\sqrt{\frac{2}{\rho AL}}\sum_{i=1}^{m}\left\{\left[f_i\cos\frac{n\pi B_i}{L}\right]\sin\frac{n\pi A_i}{L}t+\left[f_i\sin\frac{n\pi B_i}{L}\right]\cos\frac{n\pi A_i}{L}t\right\} \tag{34}$$

If we set $(i\pi A_i/L)=w_i$, the minimum energy control problem given in Eq. (23) can be recovered. The constants $K_i=f_i\cos(i\pi B_i/L)$ and $L_i=f_i\sin(i\pi B_i/L)$, $i=1,2,\ldots,m$, are to be determined for the finite time settling of the first m modes. Theoretically, the actuators move at speeds proportional to the natural frequencies of the first m controlled modes. In addition, the initial locations of the actuators depend on the finite time settling characteristics that include the final time t_f and the magnitude of the minimum energy control.

Case 1: One-Mode Control

Consider the control of the ith mode. The modal equation of the ith mode with one fixed actuator is

$$\ddot q(t)+w_i^2q_i(t)=\sqrt{\frac{2}{\rho AL}}\sin\frac{i\pi x_0}{L}[M_i\sin w_it+N_i\cos w_it] \tag{35}$$

where x_0 is the location of the fixed actuator. The corresponding modal equation of the same mode with a moving actuator is

$$\ddot q(t)+w_i^2q_i(t)=\sqrt{\frac{2}{\rho AL}}f_i\left[\cos\frac{i\pi B_i}{L}\sin w_it+\sin\frac{i\pi B_i}{L}\cos w_it\right] \tag{36}$$

If the same control saturation level and the same final time t_f are required for both fixed and moving actuators, then the following equality is a necessary condition:

$$\sqrt{M_i^2+N_i^2}\sin(i\pi x_0/L)=f_i \tag{37}$$

Note that the physical input bounds of the moving and fixed actuators are f_i and $\sqrt{M_i^2 + N_i^2}$, respectively. Thus, a moving actuator with a smaller input bound can achieve the same performance specifications as that of a fixed actuator. For both input bounds to be the same, the location of the fixed actuator should be on any antinode of the corresponding mode, i.e., $|\sin(i\pi x_0/L)| = 1$.

Case 2: Two-Mode Control

If the ith and the jth modes are to be controlled, then the two controls[5] can be stated as follows:

1) Fixed actuator

$$\ddot{q}_i(t) + w_i^2 q_i(t) = \sqrt{\frac{2}{\rho AL}}\left\{\sin\frac{i\pi x_i}{L}\sqrt{M_i^2+N_i^2}\sin(w_i t + \phi_i) + \sin\frac{i\pi x_j}{L}\sqrt{M_j^2+N_j^2}\sin(w_j t + \phi_j)\right\} \tag{38a}$$

$$\ddot{q}_j(t) + w_j^2 q_j(t) = \sqrt{\frac{2}{\rho AL}}\left\{\sin\frac{j\pi x_i}{L}\sqrt{M_i^2+N_i^2}\sin(w_i t + \phi_i) + \sin\frac{j\pi x_j}{L}\sqrt{M_j^2+N_j^2}\sin(w_j t + \phi_j)\right\} \tag{38b}$$

with

$$\phi_i = \tan^{-1}(N_i/M_i) \text{ and } \phi_j = \tan^{-1}(N_j/M_j)$$

2) Moving actuator

$$\ddot{q}_i(t) + w_i^2 q_i(t) = \sqrt{\frac{2}{\rho AL}}\left[f_i \sin\left(w_i t + \frac{i\pi B_i}{L}\right) + f_j \sin\left(w_j t + \frac{i\pi B_j}{L}\right)\right] \tag{39a}$$

$$\ddot{q}_j(t) + w_j^2 q_j(t) = \sqrt{\frac{2}{\rho AL}}\left[f_i \sin\left(w_i t + \frac{j\pi B_i}{L}\right) + f_j \sin\left(w_j t + \frac{j\pi B_j}{L}\right)\right] \tag{39b}$$

Numerical Example

Let the first and second modes be the controlled modes of a simply supported beam whose parameters are set to unity; the fixed actuators are located at $x_1 = (L/4)$ and $x_2 = (L/2)$; the natural frequencies are $w_1 = 1$ and $w_2 = 4$; the final time is $t_f = (\pi/2)$; the initial conditions are $q_1(0) = q_2(0) = \dot{q}_1(0) = \dot{q}_2(0) = 1$. After solving Eqs. (38) and (39) for $q(t)$, the boundary conditions are imposed, leading to the conditions required for finite time settling:

1) For fixed actuators:

$$\begin{bmatrix} \frac{1}{2} & \frac{4\sqrt{2}}{15} & \frac{\pi}{4} & -\frac{\sqrt{2}}{15} \\ \frac{\sqrt{2}}{15} & 0 & -\frac{\sqrt{2}}{15} & 0 \\ \frac{\pi}{4} & -\frac{4\sqrt{2}}{15} & \frac{1}{2} & -\frac{\sqrt{2}}{15} \\ -\frac{\sqrt{2}}{15} & 0 & -\frac{\sqrt{2}}{15} & 0 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \tag{40}$$

2) For moving actuators:

$$-1 = (\sqrt{2}/2)f_1\cos B_1 + [(2\sqrt{2})/3]f_2\cos(B_2/2) + [(\sqrt{2}\pi)/4]f_1\sin B_1 + (\sqrt{2}/3)f_2\sin(B_2/2) \tag{41a}$$

$$-1 = -[(\sqrt{2}\pi)/16)]f_2\cos B_2 - (\sqrt{2}/6)f_1\sin 2B_1 \tag{41b}$$

$$1 = [(\sqrt{2}\pi)/4]f_1\cos B_1 + [(2\sqrt{2})/3]f_2\cos(B_2/2) + (\sqrt{2}/2)f_1\sin B_1 - (\sqrt{2}/3)f_2\sin(B_2/2) \tag{41c}$$

$$-1 = -(\sqrt{2}/3)f_1\cos 2B_1 + [(\sqrt{2}\pi)/4]f_2\sin B_2 \tag{41d}$$

The solutions to Eqs. (40) and (41) are

$$M_1 = 0 \quad N_1 = 10.6066 \quad M_2 = -6.6505$$
$$N_2 = 72.3036 \quad f_1 = -1.1403 \quad B_1 = -1.01076$$
$$f_2 = -2.81574 \quad B_2 = 2.89421 \tag{42}$$

Note that there is a significant difference between the two control amplitudes.

VI. Control of Distributed Parameter Systems by a Single Moving Actuator

This section emphasizes the control of flexible structures by a minimum number of moving and/or fixed actuators. The objective is to control a finite set of modes, if not all, by a single actuator, which is either moving or is fixed. Recall that the control performance depends on the actuator input magnitude, as well as its location. Therefore, the location of the actuator in space and time can be used for determining control strategies that bring a system to a desired steady state in a finite time.

Consider the system described in Eqs. (6–8) with low damping ratios ζ_i, $i = 1, 2, \ldots$ and positive mass operator M. If only one actuator is desired, then the modal forces $f_i(t)$, $i = 1, 2, \ldots$, can be written as

$$f_i(t) = M\{\phi_i[P(t)]\}F(t) \tag{43}$$

Assume that the first m modes are initially active, i.e., $\eta_i(0)$ and $\dot{\eta}_i(0)$ cannot vanish simultaneously for every $1 \le i \le m$. The objective is to bring these modes to rest with one actuator, without excessively disturbing the remaining modes. First, the displacement profile $P(t)$ is kept general. The Lyapunov technique can be used to obtain the nature of the displacement $P(t)$ and magnitude $F(t)$ that satisfy the aforementioned objective.

Consider the energy function

$$V(x,\dot{x}) = \frac{1}{2}\sum_{i=1}^{m}\{w_i^2\eta_i^2 + \dot{\eta}_i^2\} \tag{44}$$

where x and $\dot{x}$ are the state vectors described by

$$x = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix}, \qquad \dot{x} = \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \vdots \\ \dot{\eta}_m \end{bmatrix} \tag{45}$$

Note that $V(x,\dot{x})$ is positive definite. The time rate of change of $V(x,\dot{x})$ is

$$\frac{dV}{dt}(x,\dot{x}) = \sum_{i=1}^{m}\dot{\eta}_i\{\ddot{\eta}_i + w_i^2\eta_i\} \tag{46}$$

Using Eq. (6), Eq. (46) becomes

$$\frac{dV}{dt}(x,\dot{x}) = \sum_{i=1}^{m}[-2\zeta_i w_i\dot{\eta}_i^2 + f_i(t)\dot{\eta}_i] \tag{47}$$

or

$$\frac{dV}{dt}(x,\dot{x}) = \sum_{i=1}^{m}\{-2\zeta_i w_i\dot{\eta}_i^2 + M[P(t)]F(t)\phi_i[P(t)]\dot{\eta}_i\} \tag{48}$$

Now the task is to find a set of smooth functions $F(t)$ and $P(t)$ such that $(\mathrm{d}V/\mathrm{d}t)\ (x,\dot{x})$ is negative definite, i.e., the total energy can be dissipated. Since the mass operator M is positive, the time rate of change of V is determined by $F(t)\ \sum_{i=1}^{m} \phi_i[P(t)]\dot{\eta}_i$.

A sufficient condition for $\dot{V}<0$ is that $F(t)\ \sum_{i=1}^{m} \phi_i[P(t)]\dot{\eta}_i$ be negative for all times t. Two ways in which this can be accomplished are given below.

1) Select $F(t) = -\sum_{i=1}^{l} \alpha_i\dot{\eta}_i$, $l \le m$ and $P(t)$, such that $\phi_i[P(t)]>0$, $i=1, 2, \ldots, k \le m$, making $F(t)\ \sum_{i=1}^{m} \phi_i[P(t)]\dot{\eta}_i$ globally negative.

2) Choose $F(t) = -\sum_{i=1}^{m} \phi_i[P(t)]\dot{\eta}_i$. In this case, the time rate of change of V is always negative, irrespective of the position profile $P(t)$ and the boundary value problem.

The case of a simply supported beam is now used to demonstrate the validity of the above development. First consider the control of a single mode with one actuator. Assume that the ith mode is controlled and is described by

$$\ddot{\eta}_i(t) + w_i^2\eta_i(t) = \sqrt{\frac{2}{\rho AL}}\, F(t) \sin\frac{i\pi x(t)}{L} \tag{49}$$

where $x(t)$ is the position of the actuator, and the system is assumed undamped. A Lyapunov energy function can be written as

$$V(\eta_i,\dot{\eta}_i) = \tfrac{1}{2}[w_i^2\eta_i^2 + \dot{\eta}_i^2] \ge 0 \tag{50}$$

and its time rate of change is

$$\frac{\mathrm{d}V}{\mathrm{d}t}(\eta_i,\dot{\eta}_i) = w_i^2\eta_i\dot{\eta}_i + \dot{\eta}_i\ddot{\eta}_i = \sqrt{\frac{2}{\rho AL}}\, F(t)\dot{\eta}_i \sin\frac{i\pi x(t)}{L} \tag{51}$$

Case 1

$$F(t) = -\alpha\dot{\eta}_i, \qquad \alpha>0 \tag{52}$$

The time rate of change of V becomes

$$\frac{\mathrm{d}V}{\mathrm{d}t}(\eta_i,\dot{\eta}_i) = -\alpha\sqrt{\frac{2}{\rho AL}}\, \dot{\eta}_i^2 \sin\frac{i\pi x(t)}{L} \tag{53}$$

For $(\mathrm{d}v/\mathrm{d}t)< 0$, we require $0< \sin\{[i\pi x(t)]/L\}<1$, or $0<x(t)<(L/i)$. Physically, this means that the actuator is forced to move between the boundary at $x = 0$ and the first nodal point corresponding to the ith mode. Note that, in this case, the location of the actuator affects the stability of the system. Simulations are shown in Figs. 11–16 for the simply

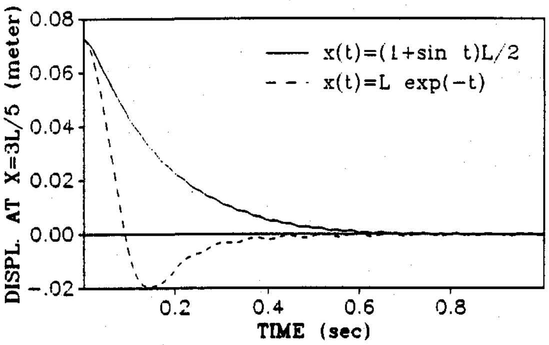

Fig. 11 Beam displacement at $x = (3L/5)$ (one actuator).

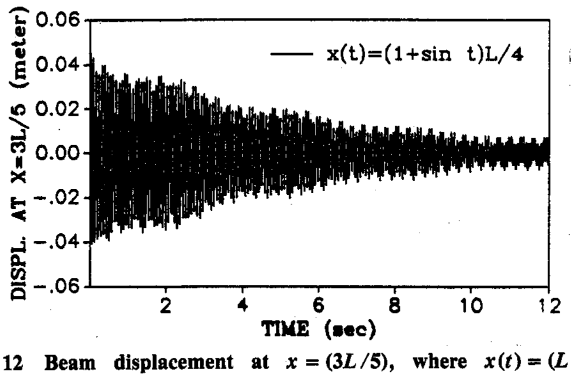

Fig. 12 Beam displacement at $x = (3L/5)$, where $x(t) = (L/4)(1 + \sin t)$.

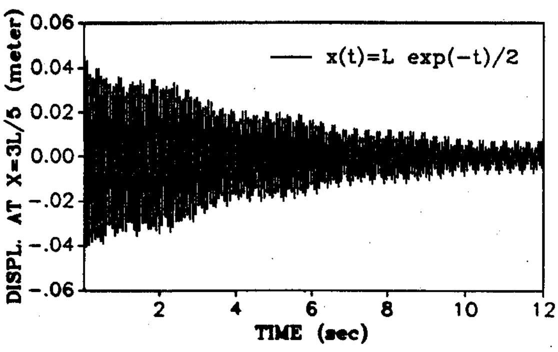

Fig. 13 Beam displacement at $x = (3L/5)$, where $x(t) = (L/2)\,Le^{-t}$.

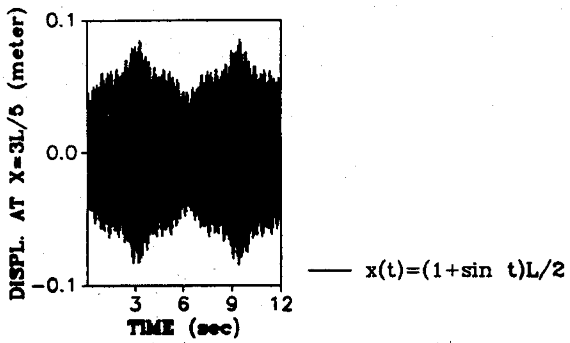

Fig. 14 Beam displacement at $x = (3L/5)$, where $x(t) = (L/2)(1 + \sin t)$.

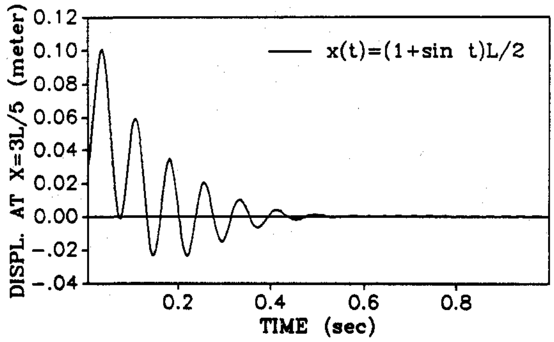

Fig. 15 Beam displacement at $x = (3L/5)$, with two initially active modes.

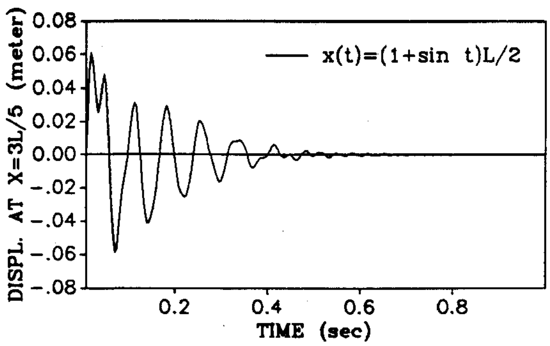

Fig. 16 Beam displacement at $x = (3L/5)$, with three initially active modes.

supported beam considered previously. In Fig. 11, the first mode is chosen as the initially active mode, with $\eta_1(0) = (L/10)$ and $\alpha = 10$. Thus, the region of travel is the whole domain of the system. The response of the beam at $x = (3L/5)$ is shown for two position profiles, $x(t) = (1 + \sin t)(L/2)$ and $x(t) = Le^{-t}$, which are bounded by 0 and L. As expected, the system can be brought to rest without excessively disturbing the higher modes. In Figs. 12–14, the second mode of vibration is selected as the initially active mode, with $\eta_2(0) = (L/2)$ and $\alpha = 0.1$. The actuator is confined to travel between 0 and $(L/2)$. In Figs. 12 and 13, the profiles $x(t) = (1 + \sin t)\ (L/4)$ and $x(t) = (L/2)\ \exp(-t)$ are selected, respectively. In this case, the gain α is chosen small so that the system can be brought to rest with no control spillover into the first mode. If the condition for the allowed region is violated, energy builds up, as shown in Fig. 14, and, therefore, it is impossible to bring the system to rest with the moving actuator. As observed in Fig. 14, energy builds up when $x(t) > (L/2)$ and decreases when $x(t) < (L/2)$.

Case 2

Assume that more than one mode is initially active. In this case, the magnitude of the control $F(t)$ is written as

$$F(t) = -\alpha\sqrt{\frac{2}{\rho AL}}\sum_{i=1}^{m}\dot{\eta}_i \sin\frac{i\pi x(t)}{L} \tag{54}$$

Simulations are shown in Figs. 15 and 16, where $\alpha = 5$. In Fig. 15, the first two modes are initially active, with $\eta_1(0) = \eta_2(0) = (L/10)$, and, thus, $m = 2$. The system is brought to rest with the actuator moving at $x(t) = (1 + \sin t)\ (L/2)$. A similar simulation is shown in Fig. 16, where the first three modes are initially active. Note that, in this case, there are no special constraints on the position profile of the actuator as long as it is confined to move within the domain of the system.

VII. Conclusions

A novel concept for the active control of distributed parameter systems based on the idea of moving force actuators was developed. It was shown possible to eliminate a predetermined set of vibratory modes by utilizing one or more force actuators moving in a special way. Two control strategies were proposed for moving actuators. The first strategy uses a finite number of distinct actuators to control an equal number of critical modes with negligible control spillover into the residual modes. The second strategy uses a single actuator to control a finite number of critical modes and is based on energy functions. Both of these strategies are fundamentally of the feedback type. However, in the first strategy, an open-loop force control can be utilized, should the initial conditions be precisely known. But since, in real systems, initial conditions are never known precisely such open loop controls must always be used in conjunction with feedback. The novel idea advanced in this work is a new direction of research for control of flexible structures.

The control of distributed parameter systems by moving actuators is certainly appealing from the point of view that the control effort can be reduced for achieving the same performance possible with a fixed actuator. The control of distributed parameter systems by one moving actuator is possible, provided that the actuator travels in restricted regions of the spatial domain in a prespecified manner.

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